A remark on the Hankel determinant formula for solutions of the Toda equation

2007 J. Phys. A: Math. Theor. 40 12661

(http://iopscience.iop.org/1751-8121/40/42/S11)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 12661–12675 doi:10.1088/1751-8113/40/42/S11

A remark on the Hankel determinant formula for solutions of the Toda equation

Kenji Kajiwara[1], **Marta Mazzocco**[2] **and Yasuhiro Ohta**[3]

[1] Graduate School of Mathematics, Kyushu University, 6-10-1 Hakozaki, Fukuoka 812-8581, Japan
[2] School of Mathematics, The University of Manchester, Sackville Street, Manchester M60 1QD, UK
[3] Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan

Received 15 January 2007, in final form 8 May 2007
Published 2 October 2007
Online at stacks.iop.org/JPhysA/40/12661

Abstract
We consider the Hankel determinant formula of the τ functions of the Toda equation. We present a relationship between the determinant formula and the auxiliary linear problem, which is characterized by a compact formula for the τ functions in the framework of the KP theory. Similar phenomena that have been observed for the Painlevé II and IV equations are recovered. The case of finite lattice is also discussed.

PACS number: 02.30.Ik
Mathematics Subject Classification: 37K10, 37K30, 34M55, 34M25, 34E05

1. Introduction

The Toda equation [27]

$$\frac{d^2 y_n}{dt^2} = e^{y_{n-1}-y_n} - e^{y_n - y_{n+1}}, \tag{1.1}$$

where $n \in \mathbb{Z}$ is one of the most important integrable systems. It can be expressed in various forms such as

$$\frac{dV_n}{dt} = V_n(I_n - I_{n+1}), \qquad \frac{dI_n}{dt} = V_{n-1} - V_n, \tag{1.2}$$

$$\frac{d\alpha_n}{dt} = \alpha_n(\beta_{n+1} - \beta_n), \qquad \frac{d\beta_n}{dt} = 2\left(\alpha_n^2 - \alpha_{n-1}^2\right), \tag{1.3}$$

where the dependent variables are related to y_n as

$$V_n = e^{y_n - y_{n+1}}, \qquad I_n = \frac{dy_n}{dt}, \qquad \alpha_n = \frac{1}{2} e^{\frac{y_n - y_{n+1}}{2}}, \qquad \beta_n = -\frac{1}{2}\frac{dy_n}{dt}. \tag{1.4}$$

The Toda equation can be reduced to the bilinear equation

$$\tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1}, \tag{1.5}$$

by the dependent variable transformation

$$y_n = \log\frac{\tau_{n-1}}{\tau_n}, \qquad V_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \qquad I_n = \frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\tau_{n-1}}{\tau_n}. \tag{1.6}$$

In general, the determinant structure of the τ function (dependent variable of bilinear equation) is the characteristic property of integrable systems. For example, the Casorati determinant formula of the N-soliton solution of the Toda equation (see, for example, [5, 28])

$$\tau_n = \mathrm{e}^{\frac{t^2}{2}}\det\left(f_{n+j-1}^{(i)}\right)_{i,j=1,\ldots,N}, \qquad f_n^{(k)} = p_k^n\,\mathrm{e}^{p_k t+\eta_{k0}} + p_k^{-n}\,\mathrm{e}^{\frac{1}{p_k}t+\xi_{k0}}, \tag{1.7}$$

where p_k, η_{k0} and ξ_{k0} $(k = 1,\ldots,N)$ are constants, is a direct consequence of the Sato theory; the solution space of soliton equations is the universal Grassmann manifold, on which infinite-dimensional Lie algebras are acting [10, 16, 28].

If we consider the Toda equation on semi-infinite or finite lattice, the soliton solutions do not exist but another determinantal solution arises. For the semi-infinite case, we impose the boundary condition as

$$\tau_{-1} = 0, \qquad \tau_0 = 1, \qquad V_0 = 0, \qquad n \geqslant 0. \tag{1.8}$$

Then τ_n admits the Hankel determinant formula [6, 7, 14]

$$\tau_n = \det(a_{i+j-2})_{i,j=1,\ldots,n}, \qquad a_0 = \tau_1, \qquad a_i = a_{i-1}', \qquad n \in \mathbb{Z}_{\geqslant 0}. \tag{1.9}$$

The important feature of this determinant formula is that the lattice site n appears as the determinant size, while for the soliton solutions the determinant size describes the number of solitons. This type of determinant formula is actually a special case of the determinant formula for the infinite lattice [13]. However, the meaning of the formula has not been yet fully understood.

The purpose of this paper is to establish a characterization of the Hankel determinant formula of the Toda equation; entries of the matrices in the determinant formula are closely related to the solution of auxiliary linear problem. Moreover, this relationship can be described by a compact formula in the framework of the theory of KP hierarchy. We note that a similar but different determinant formula for τ functions incorporating solutions of linear problems is known in the context of the Bäcklund–Darboux transformation [4] (see also the appendix).

In section 2, we discuss the Hankel determinant formula of the infinite Toda equation and present the relationship between the determinant formula and auxiliary linear problem. In section 3, we apply the results to the Painlevé II equation. We consider the case of finite lattice in section 4.

2. Hankel determinant formula of the solution of the Toda equation

2.1. Determinant formula and auxiliary linear problem

The Hankel determinant formula for τ_n satisfying the infinite Toda equation (1.5) is given by as follows.

Proposition 2.1 [13]. *For fixed $k \in \mathbb{Z}$, we have*

$$\frac{\tau_{k+n}}{\tau_k} = \begin{cases} \det\left(a_{i+j-2}^{(k)}\right)_{i,j=1,\ldots,n} & n > 0, \\ 1 & n = 0, \\ \det\left(b_{i+j-2}^{(k)}\right)_{i,j=1,\ldots,|n|} & n < 0, \end{cases} \tag{2.1}$$

$$\begin{cases} a_i^{(k)} = a_{i-1}^{(k)\prime} + \dfrac{\tau_{k-1}}{\tau_k} \displaystyle\sum_{l=0}^{i-2} a_l^{(k)} a_{i-2-l}^{(k)}, & a_0^{(k)} = \dfrac{\tau_{k+1}}{\tau_k}, \\ b_i^{(k)} = b_{i-1}^{(k)\prime} + \dfrac{\tau_{k+1}}{\tau_k} \displaystyle\sum_{l=0}^{i-2} b_l^{(k)} b_{i-2-l}^{(k)}, & b_0^{(k)} = \dfrac{\tau_{k-1}}{\tau_k}. \end{cases} \tag{2.2}$$

We shall now relate the determinant formula to the auxiliary linear problem of the Toda equation (1.2) given by

$$\begin{cases} V_{n-1}\Psi_{n-1} + I_n\Psi_n + \Psi_{n+1} = \lambda\Psi_n, \\ \dfrac{\mathrm{d}\Psi_n}{\mathrm{d}t} = V_{n-1}\Psi_{n-1}, \end{cases} \tag{2.3}$$

or

$$\begin{cases} L_n\Psi_n = \lambda\Psi_n, \\ \dfrac{\mathrm{d}\Psi_n}{\mathrm{d}t} = B_n\Psi_n, \end{cases} \tag{2.4}$$

where

$$L_n = V_{n-1}\,\mathrm{e}^{-\partial_n} + I_n + \mathrm{e}^{\partial_n}, \qquad B_n = V_{n-1}\,\mathrm{e}^{-\partial_n}. \tag{2.5}$$

The adjoint linear problem associated with the linear problem (2.3) is given by

$$\begin{cases} \Psi_{n-1}^* + I_n\Psi_n^* + V_n\Psi_{n+1}^* = \lambda\Psi_n^*, \\ \dfrac{\mathrm{d}\Psi_n^*}{\mathrm{d}t} = -V_n\Psi_{n+1}^*, \end{cases} \tag{2.6}$$

or

$$\begin{cases} L_n^*\Psi_n^* = \lambda\Psi_n^*, \\ -\dfrac{\mathrm{d}\Psi_n^*}{\mathrm{d}t} = B_n^*\Psi_n^*, \end{cases} \tag{2.7}$$

where

$$L_n^* = V_n\,\mathrm{e}^{\partial_n} + I_n + \mathrm{e}^{-\partial_n}, \qquad B_n^* = V_n\,\mathrm{e}^{\partial_n}. \tag{2.8}$$

The compatibility condition for each problem

$$\frac{\mathrm{d}L_n}{\mathrm{d}t} = [B_n, L_n], \qquad \frac{\mathrm{d}L_n^*}{\mathrm{d}t} = [-B_n^*, L_n^*] \tag{2.9}$$

yields the Toda equation (1.2), respectively.

One of our main results is that the entries of the determinants in the Hankel determinant formula arise as the coefficients of asymptotic expansions at $\lambda = \infty$ of the ratio of solutions of the linear and adjoint linear problems. To state the result more precisely, we define

$$\Xi_k(t,\lambda) = \frac{\Psi_k(t,\lambda)}{\Psi_{k+1}(t,\lambda)}, \qquad \Omega_k(t,\lambda) = \frac{\Psi_{k+1}^*(t,\lambda)}{\Psi_k^*(t,\lambda)}. \tag{2.10}$$

Theorem 2.2.

(i) The ratios $\Xi_k(t,\lambda)$ *and* $\Omega_k(t,\lambda)$ *admit two kinds of asymptotic expansions as functions of* λ *as* $\lambda\to\infty$*:*

$$\Xi_k^{(-1)}(t,\lambda) = u_{-1}\lambda^{-1} + u_{-2}\lambda^{-2} + \cdots, \tag{2.11}$$

$$\Xi_k^{(1)}(t,\lambda) = v_1\lambda + v_0 + v_{-1}\lambda^{-1} + \cdots, \tag{2.12}$$

and

$$\Omega_k^{(-1)}(t,\lambda) = u_{-1}\lambda^{-1} + u_{-2}\lambda^{-2} + \cdots, \tag{2.13}$$

$$\Omega_k^{(1)}(t,\lambda) = v_1\lambda + v_0 + v_{-1}\lambda^{-1} + \cdots, \tag{2.14}$$

respectively.

(ii) The above asymptotic expansions are related to the Hankel determinants entries $a_i^{(k)}$ *and* $b_i^{(k)}$ *as follows:*

$$\Xi_k^{(-1)}(t,\lambda) = \frac{1}{\lambda}\frac{\tau_k}{\tau_{k-1}}\sum_{i=0}^{\infty} b_i^{(k)}\lambda^{-i}, \tag{2.15}$$

$$\Xi_k^{(1)}(t,\lambda) = \frac{\tau_k^2}{\tau_{k+1}\tau_{k-1}}\left[\lambda - \frac{\left(\frac{\tau_k}{\tau_{k+1}}\right)'}{\frac{\tau_k}{\tau_{k+1}}} - \frac{1}{\lambda}\frac{\tau_k}{\tau_{k+1}}\sum_{i=0}^{\infty} a_i^{(k+1)}(-\lambda)^{-i}\right], \tag{2.16}$$

and

$$\Omega_k^{(-1)}(t,\lambda) = \frac{1}{\lambda}\frac{\tau_k}{\tau_{k+1}}\sum_{i=0}^{\infty} a_i^{(k)}(-\lambda)^{-i}, \tag{2.17}$$

$$\Omega_k^{(1)}(t,\lambda) = \frac{\tau_k^2}{\tau_{k+1}\tau_{k-1}}\left[\lambda - \frac{\left(\frac{\tau_{k-1}}{\tau_k}\right)'}{\frac{\tau_{k-1}}{\tau_k}} - \frac{1}{\lambda}\frac{\tau_k}{\tau_{k-1}}\sum_{i=0}^{\infty} b_i^{(k-1)}\lambda^{-i}\right]. \tag{2.18}$$

(iii) $\Xi_k^{(\pm1)}$ *and* $\Omega_k^{(\pm1)}$ *are related as follows:*

$$\Omega_k^{(1)}(t,\lambda)\Xi_k^{(-1)}(t,\lambda) = \frac{\tau_k^2}{\tau_{k+1}\tau_{k-1}}, \qquad \Omega_k^{(-1)}(t,\lambda)\Xi_k^{(1)}(t,\lambda) = \frac{\tau_k^2}{\tau_{k+1}\tau_{k-1}}. \tag{2.19}$$

Brief sketch of the proof of theorem 2.2. One can prove theorem 2.2 by direct calculation. From the linear problem (2.3) and (4.14), we see that $\Xi_k(t,\lambda)$ satisfies the Riccati equation

$$\frac{\partial\Xi_k}{\partial t} = -V_k\Xi_k^2 + (\lambda - I_k)\Xi_k - 1. \tag{2.20}$$

Plugging series expansion $\Xi_k = \lambda^\rho\sum_{i=0}^{\infty} h_i\lambda^{-i}$ into (2.20) and considering the balance of leading terms, we find that ρ must be $\rho = 1, -1$, which proves (2.11) and (2.12). Moreover, it is possible to verify (2.15) and (2.16) by deriving recursion relations of coefficients for each case and comparing them with (2.2). Similarly, from the Riccati equation for Ω,

$$\frac{\partial\Omega_k}{\partial t} = V_k\Omega_k^2 + (-\lambda + I_{k+1})\Omega_k + 1, \tag{2.21}$$

one can prove the statements for Ω. For (iii), putting $X_k = \frac{\tau_k^2}{\tau_{k+1}\tau_{k-1}}\frac{1}{\Xi_k^{(-1)}} = \frac{1}{V_k\Xi_k^{(-1)}}$, plugging this expression into the Riccati equation (2.21) and using (1.2), we find that X_k satisfies (2.20). Since the expansion of $\Xi_k^{(-1)}$ starts from λ^{-1}, the leading order of X_k is λ and thus $X_k = \Omega^{(1)}$. The second equation of (2.19) can be proved in a similar manner. □

2.2. KP theory

The results in the previous section can be characterized by a compact formula in terms of the language of the KP theory [10, 16, 22].

We introduce infinitely many independent variables $x = (x_1, x_2, x_3, \ldots)$, $x_1 = t$, and let $\tau_n(x)$ be the τ function of the one-dimensional Toda lattice hierarchy [10, 28] and the first modified KP hierarchy [10]. Namely, τ_n, $n \in \mathbb{Z}$, satisfy the following bilinear equations:

$$D_{x_1} p_{j+1}\left(\tfrac{1}{2}\tilde{D}\right) \tau_n \cdot \tau_n = p_j\left(\tfrac{1}{2}\tilde{D}\right) \tau_{n+1} \cdot \tau_{n-1}, \qquad j = 0, 1, 2\ldots, \tag{2.22}$$

$$\left[D_{x_1} p_j\left(\tfrac{1}{2}\tilde{D}\right) - p_{j+1}\left(\tfrac{1}{2}\tilde{D}\right) + p_{j+1}\left(-\tfrac{1}{2}\tilde{D}\right)\right] \tau_{n+1} \cdot \tau_n = 0, \qquad j = 0, 1, 2\ldots, \tag{2.23}$$

where $p_0(x), p_1(x), \ldots$ are the elementary Schur functions

$$\sum_{n=0}^{\infty} p_n(x)\kappa^n = \exp \sum_{i=1}^{\infty} x_i \kappa^i, \tag{2.24}$$

and

$$\tilde{D} = \left(D_{x_1}, \frac{1}{2}D_{x_2}, \ldots, \frac{1}{n}D_{x_n}, \ldots\right), \tag{2.25}$$

D_{x_i} $(i = 1, 2, \ldots)$ being Hirota's D-operator. Then we have the following formula.

Proposition 2.3. *For fixed $k \in \mathbb{Z}$, we have*

$$\frac{\tau_{k+n}}{\tau_k} = \begin{cases} \det\left(a_{i+j-2}^{(k)}\right)_{i,j=1,\ldots,n} & n > 0, \\ 1 & n = 0, \\ \det\left(b_{i+j-2}^{(k)}\right)_{i,j=1,\ldots,|n|} & n < 0, \end{cases} \tag{2.26}$$

where

$$a_i^{(k)} = p_i(\tilde{\partial})\frac{\tau_{k+1}}{\tau_k}, \qquad b_i^{(k)} = (-1)^i p_i(-\tilde{\partial})\frac{\tau_{k-1}}{\tau_k}, \tag{2.27}$$

and

$$\tilde{\partial} = \left(\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \ldots, \frac{1}{n}\partial_{x_n}, \ldots\right). \tag{2.28}$$

Remark 2.4. It might be interesting to remark here that $a_0^{(k)} = \frac{\tau_{k+1}}{\tau_k}$ and $b_0^{(k)} = \frac{\tau_{k-1}}{\tau_k}$ satisfy the nonlinear Schrödinger hierarchy. In fact, equations (2.2) and (2.27) with $i = 2$ imply for $a = a_0^{(k)}$ and $b = b_0^{(k)}$:

$$a_{x_2} = a_{x_1 x_1} + 2a^2 b, \qquad b_{x_2} = -\left(b_{x_1 x_1} + 2a^2 b\right). \tag{2.29}$$

Similarly, for $i = 3$, we have

$$a_{x_3} = a_{x_1 x_1 x_1} + 6aba_{x_1}, \qquad b_{x_3} = b_{x_1 x_1 x_1} + 6abb_{x_1}. \tag{2.30}$$

Here we comment that in [29], the AKNS hierarchy is analysed and it is shown that the lattice site number of one-dimensional Toda lattice is related to the Schlesinger transformation in the root lattice.

Before proceeding to the proof, we note that the auxiliary linear problem (2.3) and its adjoint problem (2.6) are recovered from the bilinear equations (2.22) and (2.23). In fact, suppose that τ_n depends on a discrete independent variable l and satisfies the discrete modified KP equation

$$D_{x_1}\tau_n(l+1)\cdot\tau_n(l) = -\frac{1}{\lambda}\tau_{n+1}(l+1)\tau_{n-1}(l), \tag{2.31}$$

$$\left(\frac{1}{\lambda}D_{x_1}+1\right)\tau_{n+1}(l+1)\cdot\tau_n(l) - \tau_n(l+1)\tau_{n+1}(l) = 0, \tag{2.32}$$

then one can show that equations (2.22) and (2.23) are equivalent to (2.31) and (2.32), respectively, through the Miwa transformation [15, 10]

$$x_n = \frac{l}{n(-\lambda)^n} \qquad \text{or} \qquad \frac{\partial}{\partial l} = -\frac{1}{\lambda}\frac{\partial}{\partial x_1} + \frac{1}{2\lambda^2}\frac{\partial}{\partial x_2} + \cdots + \frac{1}{j(-\lambda)^j}\frac{\partial}{\partial x_j} + \cdots. \tag{2.33}$$

Putting

$$\Psi_n^* = \lambda^{-n}\frac{\tau_n(l+1)}{\tau_n(l)}, \tag{2.34}$$

$$V_n = \frac{\tau_{n+1}(l)\tau_{n-1}(l)}{\tau_n(l)^2}, \qquad I_n = \frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\tau_{n-1}(l)}{\tau_n(l)}, \tag{2.35}$$

and noting $t = x_1$, the bilinear equations (2.31) and (2.32) are rewritten as

$$\begin{aligned} &\Psi_n^{*\prime} = -V_{n+1}\Psi_{n+1}^*, \\ &\Psi_n^* + I_{n+1}\Psi_{n+1}^* + V_{n+2}\Psi_{n+2}^* = \lambda\Psi_{n+1}^*, \end{aligned} \tag{2.36}$$

which are equivalent to the adjoint linear problem (2.6). Similarly, shifting $l \to l-1$ in (2.31) and (2.32) and putting

$$\Psi_{n+1} = \lambda^n\frac{\tau_n(l-1)}{\tau_n(l)}, \tag{2.37}$$

we obtain

$$\begin{aligned} &\Psi_{n+1}' = V_n\Psi_n, \\ &V_n\Psi_n + I_{n+1}\Psi_{n+1} + \Psi_{n+2} = \lambda\Psi_{n+1}, \end{aligned} \tag{2.38}$$

which is also equivalent to the linear problem (2.3).

Proof of proposition 2.3. From (2.37), (2.33) and (2.24) we have

$$\begin{aligned} \frac{\Psi_k(t,\lambda)}{\Psi_{k+1}(t,\lambda)} &= \frac{1}{\lambda}\frac{\tau_{k-1}(l-1)\tau_k(l)}{\tau_k(l-1)\tau_{k-1}(l)} = \frac{1}{\lambda}\frac{\tau_k(l)}{\tau_{k-1}(l)}\,\mathrm{e}^{-\frac{\partial}{\partial l}}\frac{\tau_{k-1}(l)}{\tau_k(l)} \\ &= \frac{1}{\lambda}\frac{\tau_k(l)}{\tau_{k-1}(l)}\exp\left(-\sum_{j=1}^{\infty}\frac{1}{j(-\lambda)^j}\frac{\partial}{\partial x_j}\right)\frac{\tau_{k-1}(l)}{\tau_k(l)} \\ &= \frac{1}{\lambda}\frac{\tau_k(l)}{\tau_{k-1}(l)}\sum_{n=0}^{\infty}p_n(-\tilde{\partial})\frac{\tau_{k-1}(l)}{\tau_k(l)}(-\lambda)^{-n}. \end{aligned}$$

Therefore equation (2.15) in theorem 2.2 implies

$$b_n^{(k)} = (-1)^n p_n(-\tilde{\partial})\frac{\tau_k(l)}{\tau_{k-1}(l)}. \tag{2.39}$$

Similarly, we have from (2.34), (2.33) and (2.24),

$$\frac{\Psi^*_{k+1}(t,\lambda)}{\Psi^*_k(t,\lambda)} = \frac{1}{\lambda}\frac{\tau_{k+1}(l+1)}{\tau_k(l+1)}\frac{\tau_k(l)}{\tau_{k+1}(l)} = \frac{1}{\lambda}\frac{\tau_k(l)}{\tau_{k+1}(l)}\,\mathrm{e}^{\frac{\partial}{\partial l}}\frac{\tau_{k+1}(l)}{\tau_k(l)}$$
$$= \frac{1}{\lambda}\frac{\tau_k(l)}{\tau_{k+1}(l)}\exp\left(\sum_{j=1}^{\infty}\frac{1}{j(-\lambda)^j}\frac{\partial}{\partial x_j}\right)\frac{\tau_{k+1}(l)}{\tau_k(l)}$$
$$= \frac{1}{\lambda}\frac{\tau_k(l)}{\tau_{k+1}(l)}\sum_{n=0}^{\infty}p_n(\tilde{\partial})\frac{\tau_{k+1}(l)}{\tau_k(l)}(-\lambda)^{-n}.$$

Therefore comparing with (2.17), we obtain

$$a_n^{(k)} = p_n(\tilde{\partial})\frac{\tau_{k+1}(l)}{\tau_k(l)}, \tag{2.40}$$

which proves proposition 2.3. □

3. Painlevé equations

3.1. Local Lax pair

Originally the relations between the determinant formula of the solutions and auxiliary linear problem have been derived for the Painlevé II and IV equations [11, 12]. In the particular case of the rational solutions of the Painlevé II and IV equations, these results give the relation between the determinant formula and the Airy function found in [3, 8]. It may be natural to regard those relationships as originating from the Toda equation, since the sequence of τ functions generated by the Bäcklund transformations of Painlevé equations is described by the Toda equation [9, 13, 23–26]. In this section, we show that the results for the Painlevé II equation can be recovered from the results in section 2. The key ingredient of the correspondence is the local Lax pair, which is the auxiliary linear problem for the Toda equation formulated by a pair of 2×2 matrices [1]:

$$\widetilde{L}_n\phi_n = \phi_{n+1}, \qquad \widetilde{L}_n(t,\lambda) = \begin{pmatrix} -I_n+\lambda & -\mathrm{e}^{-y_n} \\ \mathrm{e}^{y_n} & 0 \end{pmatrix}, \tag{3.1}$$

$$\frac{\mathrm{d}\phi_n}{\mathrm{d}t} = \widetilde{B}_n\phi_n, \qquad \widetilde{B}_n(t,\lambda) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\lambda + \begin{pmatrix} 0 & \mathrm{e}^{-y_n} \\ -\mathrm{e}^{y_{n-1}} & 0 \end{pmatrix}, \tag{3.2}$$

$$\phi_n = \begin{pmatrix} \phi_n^{(1)} \\ \phi_n^{(2)} \end{pmatrix}, \qquad y_n = \log\frac{\tau_{n-1}}{\tau_n}. \tag{3.3}$$

Similarly, the adjoint linear problem is given by

$$\widetilde{L}^*_n\phi^*_n = \phi^*_{n-1}, \qquad \widetilde{L}^*_n(t,\lambda) = \begin{pmatrix} -I_n+\lambda & \mathrm{e}^{y_n} \\ -\mathrm{e}^{-y_n} & 0 \end{pmatrix}, \tag{3.4}$$

$$\frac{\mathrm{d}\phi^*_n}{\mathrm{d}t} = \widetilde{B}^*_n\phi^*_n, \qquad \widetilde{B}^*_n(t,\lambda) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\lambda + \begin{pmatrix} 0 & \mathrm{e}^{y_n} \\ -\mathrm{e}^{-y_{n+1}} & 0 \end{pmatrix}, \tag{3.5}$$

$$\phi^*_n = \begin{pmatrix} \phi_n^{*(1)} \\ \phi_n^{*(2)} \end{pmatrix}. \tag{3.6}$$

Compatibility condition for each problem

$$\frac{\mathrm{d}\widetilde{L}_n}{\mathrm{d}t} = \widetilde{B}_{n+1}\widetilde{L}_n - \widetilde{L}_n\widetilde{B}_n, \qquad \frac{\mathrm{d}\widetilde{L}_n^*}{\mathrm{d}t} = -\widetilde{B}_{n-1}^*\widetilde{L}_n^* + \widetilde{L}_n^*\widetilde{B}_n^*, \tag{3.7}$$

gives the Toda equation (1.2), respectively. By comparing (3.1) and (3.2) with (2.3) similarly by comparing (3.4) and (3.5) with (2.6), one sees that there is a relationship between the solutions of the linear problems

$$\phi_n^{(1)} = \mathrm{e}^{-\frac{1}{2}\lambda t}\,\Psi_n, \qquad \phi_n^{(2)} = \mathrm{e}^{-\frac{1}{2}\lambda t}\,\frac{\tau_{n-2}}{\tau_{n-1}}\Psi_{n-1}, \tag{3.8}$$

$$\phi_n^{*(1)} = \mathrm{e}^{\frac{1}{2}\lambda t}\,\Psi_n^*, \qquad \phi_n^{*(2)} = -\mathrm{e}^{\frac{1}{2}\lambda t}\,\frac{\tau_{n+1}}{\tau_n}\Psi_{n+1}^*. \tag{3.9}$$

3.2. Painlevé II equation

In this section, we consider the Painlevé II equation ($\mathrm{P_{II}}$),

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = 2u^3 - 4tu + 4\left(\alpha + \frac{1}{2}\right). \tag{3.10}$$

We denote (3.10) as $\mathrm{P_{II}}[\alpha]$ when it is necessary to specify the parameter α explicitly. Suppose that τ_0 and τ_1 satisfy the bilinear equations

$$\left(D_t^2 - 2t\right)\tau_1\cdot\tau_0 = 0, \tag{3.11}$$

$$\left(D_t^3 - 2tD_t - 4\left(\alpha + \tfrac{1}{2}\right)\right)\tau_1\cdot\tau_0 = 0, \tag{3.12}$$

then it is easily verified that

$$u = \frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\tau_1}{\tau_0} \tag{3.13}$$

satisfies $\mathrm{P_{II}}[\alpha]$ (3.10). If we generate the sequence τ_n ($n\in\mathbb{Z}$) by the Toda equation

$$\tfrac{1}{2}D_t^2\tau_n\cdot\tau_n = \tau_{n+1}\tau_{n-1}, \tag{3.14}$$

then it is shown that τ_n satisfy

$$\left(D_t^2 - 2t\right)\tau_{n+1}\cdot\tau_n = 0, \tag{3.15}$$

$$\left(D_t^3 - 2tD_t - 4\left(\alpha + \tfrac{1}{2} + n\right)\right)\tau_{n+1}\cdot\tau_n = 0, \tag{3.16}$$

and that

$$u = \frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\tau_{n+1}}{\tau_n} \tag{3.17}$$

satisfies $\mathrm{P_{II}}[\alpha+n]$. In this sense, the Toda equation (3.14) describes the Bäcklund transformation of $\mathrm{P_{II}}$ (see, for example, [21]). Therefore one can apply proposition 2.1 to obtain the determinant formula: for fixed $k\in\mathbb{Z}$, we have

$$\frac{\tau_{k+n}}{\tau_k} = \begin{cases} \det\left(a_{i+j-2}^{(k)}\right)_{i,j=1,\ldots,n} & n>0,\\ 1 & n=0,\\ \det\left(b_{i+j-2}^{(k)}\right)_{i,j=1,\ldots,|n|} & n<0, \end{cases} \tag{3.18}$$

where

$$\begin{cases} a_i^{(k)} = a_{i-1}^{(k)\prime} + \dfrac{\tau_{k-1}}{\tau_k}\displaystyle\sum_{l=0}^{i-2} a_l^{(k)} a_{i-2-l}^{(k)}, & a_0^{(k)} = \dfrac{\tau_{k+1}}{\tau_k}, \\ b_i^{(k)} = b_{i-1}^{(k)\prime} + \dfrac{\tau_{k+1}}{\tau_k}\displaystyle\sum_{l=0}^{i-2} b_l^{(k)} b_{i-2-l}^{(k)}, & b_0^{(k)} = \dfrac{\tau_{k-1}}{\tau_k}. \end{cases} \tag{3.19}$$

Now consider the auxiliary linear problem for $\mathrm{P_{II}}[\alpha]$ (3.10) [9]:

$$\frac{\partial Y}{\partial \lambda} = AY, \qquad A = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & -\frac{1}{2}\frac{\tau_1}{\tau_0} \\ \frac{1}{2}\frac{\tau_{-1}}{\tau_0} & 0 \end{pmatrix}\lambda + \begin{pmatrix} -\frac{z+t}{2} & \frac{1}{2}\left(\frac{\tau_1}{\tau_0}\right)' \\ \frac{1}{2}\left(\frac{\tau_{-1}}{\tau_0}\right)' & \frac{z+t}{2} \end{pmatrix}, \tag{3.20}$$

$$\frac{\partial Y}{\partial t} = BY, \qquad B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\lambda + \begin{pmatrix} 0 & \frac{\tau_1}{\tau_0} \\ -\frac{\tau_{-1}}{\tau_0} & 0 \end{pmatrix}, \tag{3.21}$$

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad z = -\frac{\tau_1\tau_{-1}}{\tau_0^2}. \tag{3.22}$$

Comparing (3.21) with (3.2), we immediately find that

$$B = \widetilde{B_1}, \qquad Y = \phi_1. \tag{3.23}$$

We note that it is possible to regard (3.20) as the equation defining λ-flow which is consistent with evolution in t. Also, the linear equation (3.1) describes the Bäcklund transformation. Similarly, we have the adjoint problem

$$\frac{\partial Y^*}{\partial \lambda} = A^*Y^*, \qquad A^* = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & \frac{1}{2}\frac{\tau_{-1}}{\tau_0} \\ -\frac{1}{2}\frac{\tau_1}{\tau_0} & 0 \end{pmatrix}\lambda + \begin{pmatrix} -\frac{z+t}{2} & \frac{1}{2}\left(\frac{\tau_{-1}}{\tau_0}\right)' \\ \frac{1}{2}\left(\frac{\tau_1}{\tau_0}\right)' & \frac{z+t}{2} \end{pmatrix}, \tag{3.24}$$

$$\frac{\partial Y^*}{\partial t} = B^*Y^*, \qquad B^* = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\lambda + \begin{pmatrix} 0 & \frac{\tau_{-1}}{\tau_0} \\ -\frac{\tau_1}{\tau_0} & 0 \end{pmatrix}, \tag{3.25}$$

$$Y^* = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad z = -\frac{\tau_1\tau_{-1}}{\tau_0^2}, \tag{3.26}$$

where we have a correspondence

$$B^* = \widetilde{B}_0^*, \qquad Y^* = \phi_1^*. \tag{3.27}$$

Therefore, if we apply theorem 2.2 noting (3.8) and (3.9), we have the following.

Proposition 3.1. *We put*

$$\Lambda(t,\lambda) = \frac{Y_2}{Y_1}, \qquad \Pi(t,\lambda) = \frac{Y_2^*}{Y_1^*}. \tag{3.28}$$

(i) The ratios Λ and Π admit two kinds of asymptotic expansions as functions of λ as $\lambda \to \infty$

$$\Lambda^{(-1)}(t,\lambda) = u_{-1}\lambda^{-1} + u_{-2}\lambda^{-2} + \cdots, \tag{3.29}$$

$$\Lambda^{(1)}(t,\lambda) = v_1\lambda + v_0 + v_{-1}\lambda^{-1} + \cdots, \tag{3.30}$$

and

$$\Pi^{(-1)}(t,\lambda) = u_{-1}\lambda^{-1} + u_{-2}\lambda^{-2} + \cdots, \tag{3.31}$$

$$\Pi^{(1)}(t,\lambda) = v_1\lambda + v_0 + v_{-1}\lambda^{-1} + \cdots, \tag{3.32}$$

respectively.

(ii) The above asymptotic expansions are related to the Hankel determinants entries $a_i^{(k)}$ and $b_i^{(k)}$ as follows:

$$\Lambda^{(-1)}(t,\lambda) = \frac{1}{\lambda}\sum_{i=0}^{\infty} b_i^{(0)}\lambda^{-i}, \tag{3.33}$$

$$\Lambda^{(1)}(t,\lambda) = \frac{\tau_0}{\tau_1}\left[\lambda - \frac{\left(\frac{\tau_0}{\tau_1}\right)'}{\frac{\tau_0}{\tau_1}} - \frac{1}{\lambda}\frac{\tau_0}{\tau_1}\sum_{i=0}^{\infty} a_i^{(1)}(-\lambda)^{-i}\right], \tag{3.34}$$

and

$$\Pi^{(-1)}(t,\lambda) = \frac{1}{(-\lambda)}\sum_{i=0}^{\infty} a_i^{(0)}(-\lambda)^{-i}, \tag{3.35}$$

$$\Pi^{(1)}(t,\lambda) = -\frac{\tau_0}{\tau_{-1}}\left[\lambda - \frac{\left(\frac{\tau_{-1}}{\tau_0}\right)'}{\frac{\tau_{-1}}{\tau_0}} - \frac{1}{\lambda}\frac{\tau_0}{\tau_{-1}}\sum_{i=0}^{\infty} b_i^{(-1)}\lambda^{-i}\right]. \tag{3.36}$$

(iii) $\Lambda^{(\pm1)}$ and $\Pi^{(\pm1)}$ are related as follows:

$$\Pi^{(1)}(t,\lambda)\Lambda^{(-1)}(t,\lambda) = 1, \qquad \Pi^{(-1)}(t,\lambda)\Lambda^{(1)}(t,\lambda) = 1. \tag{3.37}$$

Proposition 3.1 is equivalent to the results presented in [8, 11]. In other words, the relations between determinant formula for the solution of P_{II} and auxiliary linear problems originate from the structure of the Toda equation. We also note that one can recover the results for the Painlevé IV equation [3, 12] in a similar manner.

4. Toda equation on finite lattice

4.1. Determinant formula

Let us consider the Toda equation on the finite lattice. Namely, we impose the boundary condition

$$\begin{aligned} &V_0 = 0, && V_N = 0,\\ &y_0 = -\infty, && y_{N+1} = \infty,\\ &\alpha_0 = 0, && \alpha_N = 0, \end{aligned} \tag{4.1}$$

on the Toda equation (1.1), (1.2) and (1.3), respectively. In order to realize this condition on the level of the τ function, we proceed as follows: in the bilinear equation (1.5), imposing the boundary condition on the left edge of lattice

$$\tau_{-1} = 0, \qquad \tau_0 \neq 0, \tag{4.2}$$

it immediately follows $\tau_{-2} = 0$ and one can restrict the Toda equation on the semi-infinite lattice $n \geqslant 0$. In this case, the determinant formula reduces to

$$\frac{\tau_k}{\tau_0} = \det\left(a_{i+j-2}^{(0)}\right)_{i,j=1,\cdots,k} (n \geqslant 1), \qquad a_{i+1}^{(0)} = a_i^{(0)\prime}, \qquad a_0^{(0)} = \frac{\tau_1}{\tau_0}, \tag{4.3}$$

which is equivalent to (1.9). Moreover, imposing the boundary condition on the right edge of lattice

$$\tau_N \neq 0, \qquad \tau_{N+1} = 0, \tag{4.4}$$

then we have the finite Toda equation

$$\tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1}, \qquad n = 0, \ldots, N, \qquad \tau_{-1} = \tau_{N+1} = 0. \tag{4.5}$$

It is easily verified that the boundary condition (4.4) is satisfied by putting

$$a_0^{(0)} = \sum_{i=1}^{N} c_i\, \mathrm{e}^{\mu_i t}, \tag{4.6}$$

where c_i and $\mu_i (i = 1, \ldots, N)$ are arbitrary constants.

It is sometimes convenient to consider the finite Toda equation in the form of (1.3). One reason for this is that the auxiliary linear problem associated with (1.3)

$$\alpha_{n-1}\Phi_{n-1} + \beta_n\Phi_n + \alpha_n\Phi_{n+1} = \mu\Phi_n, \qquad \frac{\mathrm{d}\Phi_n}{\mathrm{d}t} = -\alpha_{n-1}\Phi_{n-1} + \alpha_n\Phi_{n+1}, \tag{4.7}$$

or

$$L\Phi = \mu\Phi, \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}t} = B\Phi, \qquad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix}, \tag{4.8}$$

$$L = \begin{pmatrix} \beta_1 & \alpha_1 & & & \\ \alpha_1 & \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{N-2} & \beta_{N-1} & \alpha_N \\ & & 0 & \alpha_{N-1} & \beta_N \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \alpha_1 & & & \\ -\alpha_1 & 0 & \alpha_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha_{n-2} & 0 & \alpha_N \\ & & 0 & -\alpha_{N-1} & 0 \end{pmatrix}, \tag{4.9}$$

is self-adjoint [2]. The solutions of the linear problem (2.3) and adjoint linear problem (2.6) are related to Φ_n as

$$\Psi_n = \mathrm{e}^{-\mu t}(-1)^n\,\mathrm{e}^{-\frac{y_n}{2}}\,\Phi_n, \qquad \Psi_n^* = \mathrm{e}^{\mu t}(-1)^n\,\mathrm{e}^{\frac{y_n}{2}}\,\Phi_n, \qquad \mu = -\tfrac{1}{2}\lambda, \tag{4.10}$$

respectively.

Remark 4.1. The relationship between entries of determinants and the solutions of linear problems is given by applying theorem 2.2 as

$$\Omega_0^{(-1)}(t,\lambda) = \left[\frac{\Psi_1^*(t,\lambda)}{\Psi_0^*(t,\lambda)}\right]^{(-1)} = \frac{1}{\lambda}\frac{1}{a_0^{(0)}}\sum_{i=0}^{\infty} a_i^{(0)}(-\lambda)^{-i}. \tag{4.11}$$

However, it is not possible to express (4.11) in terms of the solutions of the linear problem (4.7) Φ_n by using the correspondence (4.10), since Φ_0 is not defined for the finite lattice.

4.2. Results of Moser and Nakamura

Moser [17] considered (N, N) entry of the resolvent of matrix L:

$$f(\mu) = (\mu I - L)^{-1}_{NN} = \frac{\Delta_{N-1}}{\Delta_N}, \tag{4.12}$$

where Δ_n is given by

$$\Delta_n = \begin{vmatrix} \mu-\beta_1 & -\alpha_1 & & & \\ -\alpha_1 & \mu-\beta_2 & -\alpha_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha_{n-2} & \mu-\beta_{n-1} & -\alpha_{n-1} \\ & & 0 & -\alpha_{n-1} & \mu-\beta_n \end{vmatrix}. \tag{4.13}$$

We note that $f(\mu)$ is a rational function in μ, since Δ_n is the nth degree polynomial in μ. By investigating analytic properties of $f(\mu)$, Moser derived the action-angle variables of the finite Toda equation to establish the complete integrability. Nakamura [20] further investigated the expansion of $f(\mu)$ around $\mu = \infty$ to obtain

$$f(\mu) = \frac{\Delta_{N-1}}{\Delta_N} = \frac{1}{\mu}\frac{1}{g_0}\sum_{i=0}^{\infty} g_i(-2\mu)^{-i}, \qquad g_i' = g_{i+1}, \tag{4.14}$$

and claimed that g_i are the entries of the determinant formula (4.3), which is quite similar to our result. Let us discuss this result from our point of view.

By expanding the determinant in (4.13) with respect to the nth row, we have the recurrence relation of Δ_n:

$$\Delta_n = (\mu - \beta_n)\Delta_{n-1} - \alpha_{n-1}^2\Delta_{n-2}. \tag{4.15}$$

Also, one can show by induction

$$\Delta_n' = -2\alpha_n^2\Delta_{n-1}. \tag{4.16}$$

Comparing (4.15) and (4.16) with the linear problems (2.3) and (4.7), we have from (1.4) and (4.10)

$$\Delta_n = (-2)^{-n}\Psi_{n+1} = 2^{-n}\,\mathrm{e}^{-\frac{y_n}{2}}\,\Phi_{n+1}. \tag{4.17}$$

Now proposition 2.1 and theorem 2.2 with $k = N$ yield

$$\begin{gathered} \frac{\tau_{N-n}}{\tau_N} = \det\big(b_{i+j-2}^{(N)}\big)_{i,j=1,\ldots,n}, \\ b_i^{(N)} = b_{i-1}^{(N)\prime} + \frac{\tau_{N+1}}{\tau_N}\sum_{j=0}^{i-2} b_j^{(N)} b_{i-2-j}^{(N)}, \qquad b_0^{(N)} = \frac{\tau_{N-1}}{\tau_N}, \\ \left[\frac{\Psi_N(t,\lambda)}{\Psi_{N+1}(t,\lambda)}\right]^{(-1)} = \frac{1}{\lambda}\frac{\tau_N}{\tau_{N-1}}\sum_{i=0}^{\infty} b_i^{(N)}\lambda^{-i}. \end{gathered} \tag{4.18}$$

Then taking the boundary condition (4.4) into account, noting that Δ_n is polynomial of degree n in $\mu = -\frac{\lambda}{2}$, equation (4.18) can be rewritten by using (4.17) as

$$\begin{aligned} \frac{\Delta_{N-1}}{\Delta_N} &= \frac{1}{\mu}\frac{\tau_N}{\tau_{N-1}}\sum_{i=0}^{\infty} b_i^{(N)}(-2\mu)^{-i}, \qquad & b_i^{(N)} &= b_{i-1}^{(N)\prime}, \\ \frac{\tau_{N-n}}{\tau_N} &= \det\big(b_{i+j-2}^{(N)}\big)_{i,j=1,\ldots,n}, & b_0^{(N)} &= \tfrac{\tau_{N-1}}{\tau_N}, \end{aligned} \tag{4.19}$$

which is nothing but (4.14). In order to satisfy the boundary condition (4.2) at the left edge of lattice, we choose $b_0^{(N)}$ to be sum of N terms of the exponential function.

In summary, Nakamura's result may be interpreted as the determinant formula viewed from the opposite direction of the lattice. Namely, starting from $n = N$ under normalization $\tau_N = 1$, it describes such formula that expresses τ_{N-n} in terms of an $n \times n$ determinant. Since τ function of the finite Toda equation is invariant with respect to inversion of the lattice $(n \to N-n)$, it is also possible to regard this formula as expressing τ_n as the $n\times n$ determinant under the normalization $\tau_0 = 1$. Also, it should be remarked that the resolvent of L appeared because Δ_n, the principal minor determinant of $\mu I - L$, satisfies the auxiliary linear problem of the finite Toda equation.

Remark 4.2. In order to obtain a 'normal' determinant formula, we may consider (1, 1) entry of the resolvent of L:

$$g(\mu) = (\mu I - L)^{-1}_{11} = \frac{\overline{\Delta}_1}{\overline{\Delta}_0}, \tag{4.20}$$

$$\overline{\Delta}_n = \begin{vmatrix} \mu-\beta_{n+1} & -\alpha_{n+1} & & & & \\ -\alpha_{n+1} & \mu-\beta_{n+2} & -\alpha_{n+2} & & & \\ & \ddots & \ddots & \ddots & & \\ & & -\alpha_{N-2} & \mu-\beta_{N-1} & -\alpha_{N-1} \\ & & 0 & -\alpha_{N-1} & \mu-\beta_N \end{vmatrix}. \tag{4.21}$$

The recurrence relations for $\overline{\Delta}_n$ are given by

$$\overline{\Delta}_n = (\mu-\beta_{n+1})\overline{\Delta}_{n+1} - \alpha_{n+2}^2\overline{\Delta}_{n+2}, \qquad \overline{\Delta}_n' = 2\alpha_n^2\overline{\Delta}_{n+1}, \tag{4.22}$$

which implies

$$\overline{\Delta}_n = (-2)^n\Psi_n^* = 2^n\,\mathrm{e}^{\frac{y_n}{2}}\,\Phi_n. \tag{4.23}$$

Therefore proposition 2.1 and theorem 2.2 yield

$$\begin{aligned} &\frac{\overline{\Delta}_1}{\overline{\Delta}_0} = \frac{1}{\mu}\frac{\tau_0}{\tau_1}\sum_{i=0}^{\infty} a_i^{(0)}(2\mu)^{-i}, \qquad a_i^{(0)} = a_{i-1}^{(0)\prime}, \qquad a_0^{(0)} = \frac{\tau_1}{\tau_0}, \\ &\frac{\tau_n}{\tau_0} = \det\left(a_{i+j-2}^{(0)}\right)_{i,j=1,\ldots,n}. \end{aligned} \tag{4.24}$$

5. Concluding remarks

In this paper, we have established the relationship between the Hankel determinant formula and the auxiliary linear problem. We have also presented a compact formula of the τ function in the framework of the KP theory. The similar phenomena that have been observed in the Painlevé II and IV equations can be recovered from this result. We have also pointed out that Moser and Nakamura's result on the finite Toda equation can be understood naturally in our framework.

Since the Toda equation can be seen in various contexts, we expect that the structure presented in this paper can be observed in wide area of physical and mathematical sciences. Moreover, it might be an intriguing problem to study whether similar phenomenon can be observed or not for the periodic lattice, where the theta functions play the role of the τ functions.

Acknowledgments

The authors would like to express their sincere thanks to N Joshi for fruitful discussions. They thank Y Nakamura and G Carlet for helpful conversation. They also thank anonymous referees for drawing their attention to [4, 29]. This research was partially supported by MIMS, the Manchester Institute for Mathematical Sciences. KK is supported by the JSPS grant-in-aid for Scientific Research (B)15340057 and (A)16204007. He also acknowledges the support by the 21st Century COE program at the Faculty of Mathematics, Kyushu University.

Appendix

In this appendix, we explain the importance of boundary conditions for soliton equations. In the case of the Toda equation (1.5), two types of determinant expressions of solutions are known according to the boundary conditions, the Casorati determinant (1.7) for the infinite lattice and the Hankel determinant (1.9) for the semi-infinite lattice. In principle, the infinite and semi-infinite lattices are equivalent in the following sense: the semi-infinite one is obtained from the infinite one by applying the boundary condition (1.8), and the infinite one is recovered from the semi-infinite one by taking the limit of the boundary going away to infinity. However, these derivations of one from another are not compatible with the determinant structure of solutions. For example, the solutions of semi-infinite one which survive in the infinite lattice limit were obtained only by giving up the Hankel determinant structure [18, 19].

Similar discrepancy of solutions is observed in various soliton equations. For instance, it is well known that the nonlinear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} + \epsilon|u|^2 u = 0, \qquad \epsilon = \pm 1,$$

has two types of solutions according to the sign of parameter ϵ. In the focusing case ($\epsilon = +1$), the above equation with the boundary condition $|u| \to 0$ as $x \to \pm\infty$ admits the bright soliton solutions which are written in terms of the Hankel determinant (1.9), while in the case of defocusing parameter ($\epsilon = -1$), we have the dark soliton solutions for the boundary condition $|u| \to$ (positive constant) as $x \to \pm\infty$, which are expressed by the determinant of type (1.7). Concerning practical expressions of solutions, it is necessary to use different types of determinants for different boundary conditions, and they are not transformed to each other.

The other Hankel determinant expression (2.1) of solutions for the infinite Toda lattice is a generalization of (1.9). It should be pointed out that the entries of the determinants are given as differential polynomials of seed functions $a_0^{(k)}$ and $b_0^{(k)}$. This Hankel determinant might be a link between the two types (1.7) and (1.9). We remark that in the theory of the Bäcklund–Darboux transformation, the ratio of τ functions is expressed as the Wronskian determinant of the eigenfunctions for the associated linear problem. The essential difference of the Hankel determinant (2.1) from the Wronskian expression originates from the quadratic terms in the recursion relation (2.2). For instance, a formula, analogous to (2.1) expressing the ratio of the τ functions in different sites, appeared in [4] within the context of the Sato–Segal–Wilson Grassmannian, in which they started from the more general Lax operator than that of the Toda equation itself. If we simply apply the suitable reduction to the one-dimensional Toda equation without breaking the Wronskian structure, then the formula in [4] reduces to the Hankel determinant formula (1.9) for the semi-infinite lattice. The quadratic terms in the recursion relation are not recovered straightforwardly since the entries of Wronskian are recursively determined by differentiation without quadratic terms. On the other hand, the Wronskian expression of Darboux transformation gives the general solution including the case of the infinite Toda lattice. This implies that as determinant expressions, the Hankel one and Wronskian one do not directly correspond entry by entry up to row and column operations, while as a function, they may correspond and in order to establish exact correspondence, it seems necessary to expand the Wronskian and recast it into the Hankel form. The formal framework in [4] is quite general; thus studying the derivation of (2.1) from the general theoretical viewpoint deserves further investigation in order to clarify the relation between determinant expressions of solutions for the infinite and semi-infinite lattices.

References

[1] Fadeev L D, Takhtajan L A and Tarasov V O 1983 Local Hamiltonians for integrable quantum models on a lattice *Teor. Mat. Fiz.* **57** 163–81

[2] Flaschka H 1974 The Toda lattice: I. Existence of integrals *Phys. Rev.* B **9** 1924–5

[3] Goto H and Kajiwara K 2005 Generating function related to the Okamoto polynomials for the Painlevé IV equation *Bull. Aust. Math. Soc.* **71** 517–26

[4] Helminck G F and van de Leur J W 2001 Geometric Bäcklund–Darboux transformations for the KP hierarchy *Publ. Res. Inst. Math. Sci.* **37** 479–519

[5] Hirota R, Ito M and Kako F 1988 Two-dimensional Toda lattice equations *Prog. Theor. Phys. Suppl.* **94** 42–58

[6] Hirota R 1988 Toda molecule equations *Algebraic Analysis* vol 1 (Boston: Academic) p 203–16

[7] Hirota R 1987 Discrete two-dimensional Toda molecule equation *J. Phys. Soc. Japan* **56** 4285–8

[8] Iwasaki K, Kajiwara K and Nakamura T 2002 Generating function associated with the rational solutions of the Painlevé II equation *J. Phys. A: Math. Gen.* **35** L207–11

[9] Jimbo M and Miwa T 1981 Monodoromy preserving deformation of linear ordinary differential equations with rational coefficients: II *Physica* D **2** 407–48

[10] Jimbo M and Miwa T 1983 Solitons and infinite dimensional Lie algebras *Publ. Res. Inst. Math. Sci.* **19** 943–1001

[11] Joshi N, Kajiwara K and Mazzocco M 2004 Generating function associated with the determinant formula for solutions of the Painlevé II equation *Astérisque* **274** 67–78

[12] Joshi N, Kajiwara K and Mazzocco M 2006 Generating function associated with the determinant formula for solutions of the Painlevé IV equation *Funkcial. Ekvac* **49** 451–68

[13] Kajiwara K, Masuda T, Noumi M, Ohta Y and Yamada Y 2001 Determinant formulas for the Toda and discrete Toda equations *Funkcial. Ekvac.* **44** 291–307

[14] Leznov A N and Saveliev M V 1981 Theory of group representations and integration of nonlinear systems $x_{a,z\bar{z}} = \exp(kx)_a$ *Physica* D **3** 62–72

[15] Miwa T 1982 On Hirota's difference equations *Proc. J. Acad.* A **58** 9–12

[16] Miwa T, Jimbo M and Date E 2000 SOLITONS *Differential Equations Symmetries and Infinite Dimensional Algebras (Cambridge Tracts in Mathematics* vol 135*)* (Cambridge: Cambridge University Press)

[17] Moser J 1975 Finitely many mass points on the line under the influence of an exponential potential—an integrable system *Dynamical Systems, Theory and Applications (Lecture Notes in Physics* vol 38*)* ed J Moser (Berlin: Springer) p 467–97

[18] Nakamura A 1990 Explicit one-soliton solutions of the 1+1 dimensional Toda molecule equation and their large molecule limit of the infinite Toda lattice *J. Phys. Soc. Japan* **59** 4272–8

[19] Nakamura A 1998 Explicit N-soliton solutions of the 1+1 dimensional Toda molecule equation *J. Phys. Soc. Japan* **67** 791–8

[20] Nakamura Y 1994 A tau-function of the finite nonperiodic Toda lattice *Phys. Lett.* A **195** 346–50

[21] Noumi M 2004 *Painlevé Equations Through Symmetry (Translations of Mathematical Monographs* vol 223*)* (Providence, RI: American Mathematical Society)

[22] Ohta Y, Satsuma J, Takahashi D and Tokihiro T 1988 An elementary introduction to Sato theory *Prog. Theor. Phys.* **80** 210–41

[23] Okamoto K 1987 Studies on the Painlevé equations: I. Sixth Painlevé equation P_{VI} *Ann. Math. Pure Appl.* **146** 337–81

[24] Okamoto K 1987 Studies on the Painlevé equations: II. Fifth Painlevé equation P_{V} *Japan. J. Math.* **13** 47–76

[25] Okamoto K 1986 Studies on the Painlevé equations: III. Second and fourth Painlevé equations, P_{II} and P_{IV} *Math. Ann.* **275** 221–55

[26] Okamoto K 1987 Studies on the Painlevé equations: IV. Third Painlevé equation P_{III} *Funkcial. Ekvac.* **30** 305–32

[27] Toda M 1967 Vibration of a Chain with Nonlinear Interaction *J. Phys. Soc. Japan* **22** 431–6

[28] Ueno K and Takasaki K 1984 Toda lattice hierarchy *Group Representations and Systems of Differential Equations (Advanced Studies in Pure Mathematics* vol 4*)* (Amsterdam: North-Holland) p 1–95

[29] van de Leur J 1998 Schlesinger–Bäcklund transformations for the N-component KP *J. Math. Phys.* **39** 2833–47